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Using Circle-and-Square Intersections to Engage Students in the Process of Doing Geometry

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WHAT does it mean to "do geometry"? Geometry is often presented to students as a finished product; they learn a preset system of definitions, theorems, constructions, and proofs. Although students need to learn about geometric knowledge that has been developed over the course of the human journey, they also need to learn that "geometry is not so much a branch of mathematics as a way of thinking" (Atiyah 1982). In essence, for students to really do geometry, they must develop and use geometric habits of mind (Goldenberg, Cuoco, and Mark 1998). (See also Driscoll et al. [2009] regarding habits of mind.) That is, they need to engage actively in open-ended problems that allow them to define mathematical objects and discover their properties. (See also de Villiers, Govender, and Patterson [2009] regarding definition.) Furthermore, they need to explore and make conjectures about relationships, develop explanatory proofs, and create classification schemes to systematize their understanding. Indeed, by highlighting the importance of both content and process standards, the National Council of Teachers of Mathematics (NCTM) has actively promoted the engagement of students in geometric thinking in the broad sense described above (NCTM 2000).

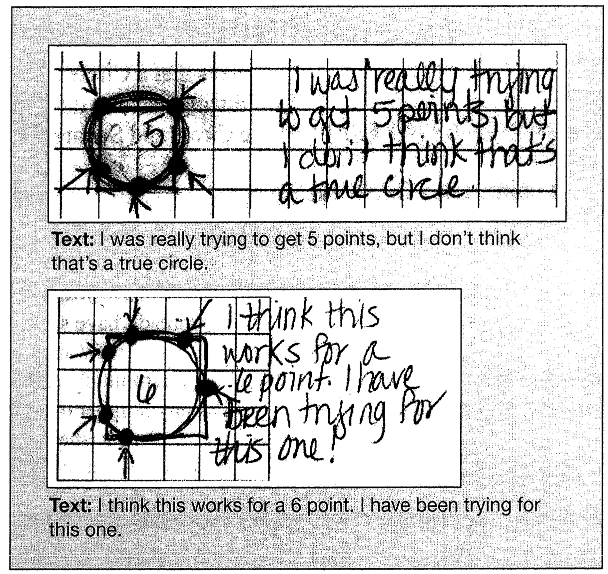
Given the need for students to engage actively in doing geometry, the challenge falls on teachers to develop, implement, and refine worthwhile tasks to use with students (NCTM 1991). An example of such a rich, open-ended task is given in this article. The story presented is a reflection of the authors' orchestration of this task in our own classes with high school students, preservice elementary school teachers, and in-service middle and high school teachers in Washington, Oregon, and Michigan over the past few years. In our article in the December 2006–January 2007 *Mathematics Teacher,* we introduced this task to readers and discussed how it could be used to promote meaningful discussion and group work (Canada and Blair 2006/2007). In this article, we focus on how we used this task to engage students in doing geometry in the spirit of authentic mathematical research, namely, as a cyclic process of conjecture, refutation, definition, and proof (Lakatos 1975). We offer our ideas and reflections not as an activity to be used in any particular classroom but as an example of a task that mathematics educators at all levels can use to consider some important issues in trying to engage students in doing geometry.

How Many Points of Intersection Are Possible between a Circle and a Square?

Our investigation with students always began with this simple, open-ended question, which we encouraged students to consider in small groups using pencil and paper. Students at both the high school and college levels generally began exploring the situation and made freehand drawings that convinced most of them that zero, one, two, four, and eight points of intersection are possible for a circle and square lying in the same plane. Some students produced drawings with five, six, or seven intersection points, but they were often unable to convince their peers because of the imprecision of their drawings. Indeed, some students became convinced that certain numbers of points were not possible, concluding, for example, "You can't get five unless you change the square to a rectangle or the circle to an oval" (Canada and Blair 2006 /2007). After about half an hour of small-group exploration, we facilitated a whole-group discussion in which students presented and compared ideas based on their freehand sketches. The goal was for students to begin answering the question while also deciding as a group that they needed more-precise representational tools to explore the existence of certain possibilities. Another point highlighted during the discussion was the fact that the lack of a counterexample does not guarantee the truth of a conjecture.

The fact that their drawings were imprecise helped them realize that they needed to justify why a given situation was impossible.

Once students agreed on the need for more-precise representational tools, they were provided with graph paper, compasses, and rulers. Some classes were also able to use The Geometer's Sketchpad (Jackiw 2001) to explore the situation. These tools enabled students to make more-precise drawings, as illustrated in one preservice teacher's work (fig. 19.1), and they were eventually able to use these tools to convince one another that all the numbers of intersection points from zero to eight were possible.



Text: I was really trying to get 5 points, but I don't think that's a true circle.

Text: I think this works for a 6 point. I have been trying for this one.

Fig. 19.1. Initial investigation of five- and six-point intersections

The level of discourse during this part of the investigation was kept informal for the task to be engaging for all students. Students were encouraged to refer to their existing knowledge when explaining their ideas but were not forced to present ideas in a formal way. We facilitated students' work at this stage mainly by asking clarifying and probing questions and by refraining from judging students' work, deferring questions such as "Is this correct?" to the small groups. We took this perspective so that students would take ownership of the activity and see themselves as geometers. We also wanted them to see an increasing role for rigor as their exploration of the material matured.

In What Different Ways Can a Circle and a Square Intersect?

The richness of this investigation lies not only in determining that each number of intersection points from zero to eight is possible but also in classifying the different ways in which each number of points can be attained. As students explored the initial question concerning the number of intersection points, they noticed that a configuration for a given number of intersection points could be done in multiple ways. When we asked students to describe the different ways they found, the focus of the investigation shifted to the geometric process of defining. Both high school and college students believed that the type of intersection point was rele-vant to whether two ways were the same, and most groups began to distinguish among intersections at a vertex, tangent to a side, and crossing through a side. Some groups of students also believed that the relative size and position of the circle to the square was also important, and they generally added descriptions to distinguish whether one was inside or "mostly" inside the other, as illustrated in figure 19.2.

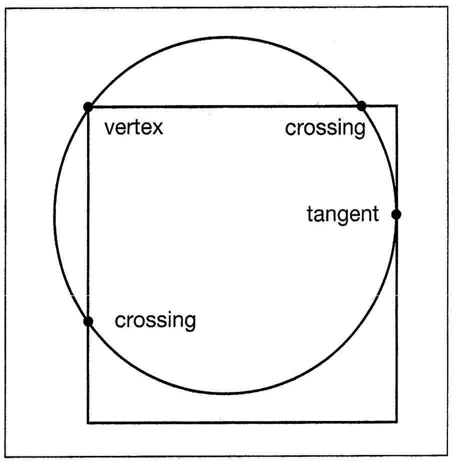


Fig. 19.2. An example of a vertex, a tangent, and two crossing intersection points with the circle "mostly" inside the square

An important aspect to note is that as students decided how they would describe different configurations, they also began to classify the different possibilities. Thus, the geometric processes of defining and classifying emerged together. As they worked, students soon realized that some of their methods were imprecise and also that the more complex their descriptions, the more difficult was

the task of organizing all the possible configurations. For this reason, the more experienced groups of students generally decided not to include the relative position descriptions, and limited their system to the number and type of intersection points. Eventually, the students decided that different solutions should be based solely on the defining set of properties and no others, which is the essence of a precise mathematical definition. Thus, for example, all three of the configurations in figure 19.3 are different if relative position is included in the definition and the same if it is not included.

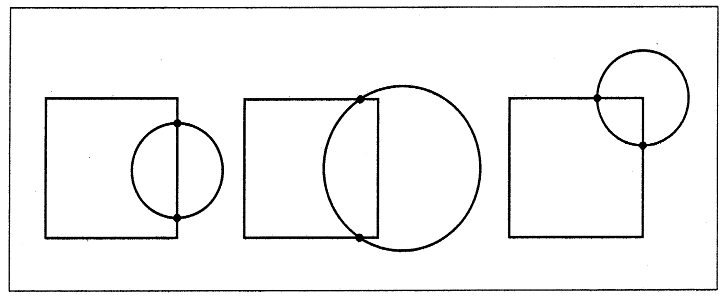


Fig. 19.3. Three configurations classified as having two crossing intersection points

Indeed, as they generated and compared different sketches, the students often noted the difficulty of ignoring attributes that were not part of their agreed-on defining properties. They began to view the subject more formally, by considering consequences of their stipulated definitions. This perspective has been characterized as an important aspect of advanced mathematical thinking (Tall 1992).

What Relationships Do You Notice in Your Classification?

As students worked to classify the different ways a circle and square can intersect according to their definitions, we encouraged them to look for connections and relationships. In one class of preservice and in-service teachers, several students noticed that the total number of crossing intersection points seemed to be even. When asked to share their ideas in a whole-class discussion, the group offered this relationship as a conjecture that they believed would always be true. The entire class was then asked to consider this relationship. Was it always true? Could we explain why or why not? After a few moments, several students found counterexamples to the conjecture, as shown in figure 19.4. They found that by having the circle "cross" at a vertex, the number of crossing points, which they had defined as crossing between endpoints of a side, could be odd.

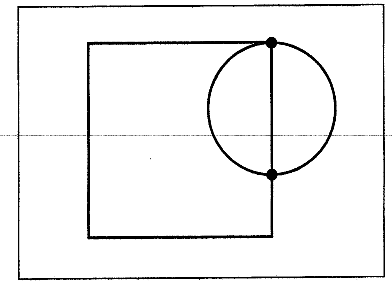


Fig. 19.4. A counterexample to the "even number of crossing points" conjecture because one of the intersection points crosses at a vertex

What happened next highlights just how this investigation really engaged the students in doing geometry. The students decided that the relationship, namely, that configurations always have an even number of crossing points, was really true but that their definitions and classification scheme were inadequate to support this relationship. At this point the instructors encouraged the students to discuss with one another which particular features were important and not to be afraid to "think outside the box." Several students noted that not all the "vertex" points were the same; whether the circle "crossed over" the related segment (an aspect they had already deemed relevant for side points) also made a difference. After some debate, the students decided that they needed to change their definitions for this relationship to be true. They decided that the type of intersections really exhibited two independent characteristics; *where* they occurred (at a vertex or along the interior of a side) and *how* they occurred (tangent to the point or crossing through the point). Thus, they created a classification that yielded the four combinations of these characteristics, as shown in figure 19.5.

With their new classification scheme, students were able to justify why every configuration must have an even number of crossing points. They also agreed that the new scheme was more elegant than their original scheme. We, as instructors, were filled with a sense of joy and amazement by the fact that our students' journey had truly become one of mathematical creation. Lakatos (1975), in his seminal book *Proof and Refutations*, noted that mathematics grows not in the order presented in textbooks but rather through a recurring cycle in which relationships are discovered, proofs are proposed and refuted, and then relationships are ultimately proved by refining appropriate arguments and definitions (Clements 2003). Our students had engaged in this cycle, from making initial definitions based on experimentation, to noticing an important relationship, to the final refinement of their scheme that enabled them to prove the result they felt should be true. Their agreed-on final proof used the fact that because the square is a simple, closed

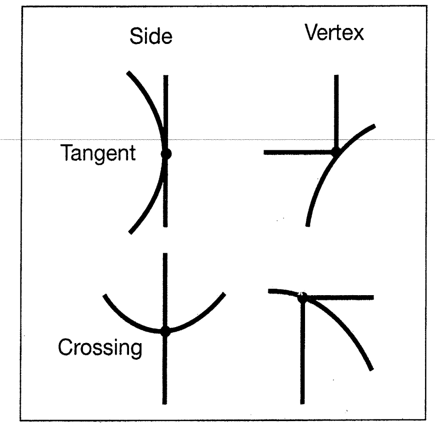


Fig. 19.5. Students' final classification scheme

curve, it partitioned the plane into points interior to, on, and exterior to the square. They argued that by starting at a nonintersection point on the circle and moving clockwise around it one time, they would move through each intersection point once. As they did so their location "state" from interior to exterior, or vice versa, would change only once at crossing intersection points, which is why they had to include crossing vertex points! Hence, because the state change toggled between two options (interior and exterior) and the final state had to be the same as the original state, they concluded that the number of crossing points had to be even.

Once the students were satisfied with their proof, we probed their understanding by trying to uncover some hidden assumptions concerning continuity and finiteness of the total number of intersection points. They argued why they knew the total number of intersection points had to be finite, but did not really understand the need for continuity. We noted that they were in good company, that in fact Euclid made a similar assumption in Proposition 1, Book 1 of *Elements,* and that only relatively recently Hilbert and others saw the need for a formal axiom of continuity (Hilbert 2001). The students were excited to know that their mathematical activity paralleled that of historical figures. After some discussion they agreed on the need to guarantee that curves are continuous to avoid the situation of one curve failing to intersect another by passing through a "microscopic hole." It also made sense to them that we, following in the footsteps of Hilbert, should take this property as an axiom. This follow-up discussion had the additional benefit of helping students articulate the difference between a definition and an axiom, two concepts sometimes viewed identically as "things taken to be true" even though they have very different roles in the process of doing geometry.

Once students settled on their classification scheme, they made a systematic search for all the different configurations for a given number of intersection points. Indeed, their theorem concerning an even number of crossing points became useful at this point because they could use it to narrow down the list. For example, for five total intersection points, zero, two, or four of them must be crossing points. Zero can be quickly eliminated as impossible, leaving only two and four. From the remaining possibilities, students determined that four configurations are possible, as shown in figure 19.6. Note that the first pair of configurations in figure 19.6 has two crossing points and the second pair has four crossing points.

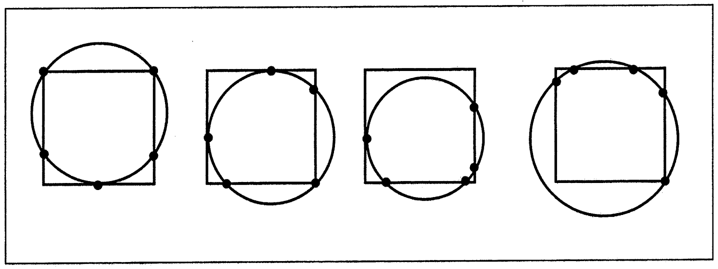


Fig. 19.6. Possible five-point intersections

How Can You Construct a Given Configuration?

As students worked to classify all possible configurations, they continued to identify and explore interesting relationships and soon believed that their classification was complete. When they reached this juncture, we shifted the investigation one more time and asked students how to construct (as in the Euclidean sense, using the equivalent of a straightedge and compass) some of the configurations they had found. This extension was not completely out of the blue for students, because they had already been trying to represent different configurations using a variety of construction tools. The problem for most students, however, lies in moving from using the tools to make an approximate drawing to using them in a coordinated way that logically results in a precise construction (Clements 2003). We asked students for an explicit sequence of steps, that is, an algorithm, that would produce the configuration. Such statements as "… and then play with the circle until it fits" were not allowed (Canada and Blair 2006/2007). Our reason for this assignment was to bridge the gap between finding a configuration and formally proving that it actually exists; we intended for students' constructions to lead to constructive proofs (de Villiers 2003).

The students found these constructions to be both interesting and challenging. As they worked on a variety of constructions, students began to notice and

use different relationships about circles, such as the fact that the perpendicular bisector of a chord intersects the center of a circle. Teachers tend to assume that if students are aware of a given theorem, they will be able to use it effectively, but we found that our students struggled to see how a theorem might apply to a particular situation. As they tried to construct different configurations, however, students got quite creative. By using probing questions, we were able to help them draw out and identify more-general relationships. For example, one student discovered that he could construct a "corner" of a square within a given circle by connecting a point on the circle to the endpoints of any diameter (fig. 19.7). This technique was useful for creating several configurations and was quickly adopted by many students. When asked why the strategy worked, the class discussed it for more than ten minutes before they recognized it as a special case of the inscribed angle theorem, namely, that the measure of an inscribed angle of a circle equals one-half the measure of its corresponding central angle.

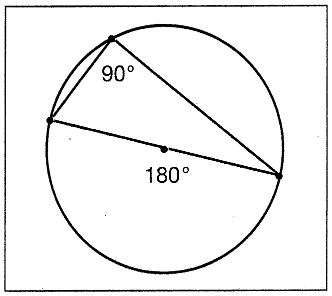


Fig. 19.7. Constructing a right angle within a circle as a useful application of the inscribed angle theorem

An interesting aspect of having students construct different configurations is the variety of ways in which they may approach the task. When a theorem or construction is presented in a textbook, it is often justified in only one way. Thus, students tend to think that a theorem has only one proof and furthermore, that someone has already found it. The constructions in this investigation were different in that students could not copy them from a book. Instead, they tended to find several ways to construct and justify a given configuration. Even more surprising was the fact that the students' methods were correct but often substantially different from what we had anticipated they would find. For example, consider the five-point configuration with one tangent-side, two crossing-side, and two tangent-vertex intersection points (shown in fig. 19.8). Starting with a square, we correctly anticipated that students would first construct the perpendicular bisector of one side, *AE*, making the endpoints the two tangent-vertex points and noting that the perpendicular bisector would intersect the opposite side at the tangent-side point (*C*). From that point on, however, none of the students finished the construction in the anticipated way. We had expected students to construct the chord from the tangent-side point (*C*) to one of the two tangent-vertex points (*E*) and then construct its perpendicular bisector. Since perpendicular bisectors of chords of a circle contain the center of the circle, we can determine the needed center by constructing the intersection of the perpendicular bisectors. Once we have located the center, we can construct the circle with that center through any of the other three points, as shown in figure 19.8.

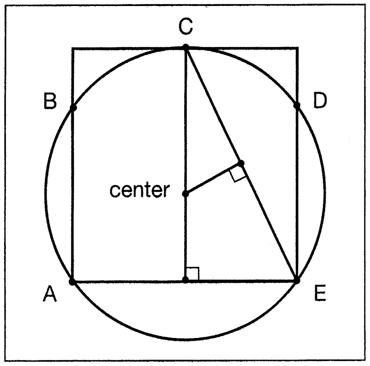


Fig. 19.8. An anticipated construction

The students' actual approaches, however, were more complex and brought out some interesting relationships. One group used cyclic quadrilaterals, a topic we had recently explored. They constructed the segment from one of the tangent-vertex points *A* to the tangent-side point *C* and noticed that if that were connected to the appropriate point *D*, one of the desired crossing-side points, then a cyclic quadrilateral *ACDE* could be formed, as shown in figure 19.9. Furthermore, they remembered that opposite angles of a cyclic quadrilateral are supplementary: if one angle is 90 degrees, the one opposite is also 90 degrees. Hence, they needed to make angle *ACD* a right angle, so they constructed the perpendicular at *C*. They then reasoned that segment *DA* had to be a diameter using the inscribed angle theorem. The midpoint *M* of this diameter determined the center of the circle.

Another group of students came up with a totally different approach. After an extended amount of exploration using The Geometer's Sketchpad, they created the following directions. First construct the midpoints *C* and *F* of the top and bottom sides of the square. Then find the midpoint *G* of segment *CF.* Next find the midpoint *H* of segment *GF*. Finally, find the midpoint *M* of segment *GH;* this point is the desired center of the circle. Crazy as it seems, their method seemed to work, but they were not sure why (fig. 19.10).

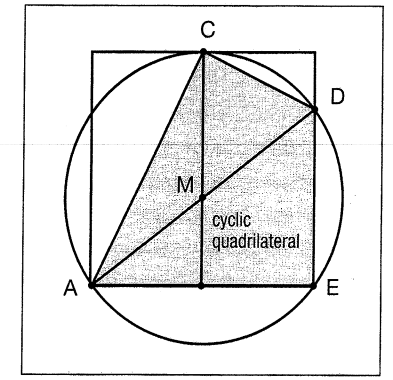


Fig. 19.9. One construction by students of a five-point intersection

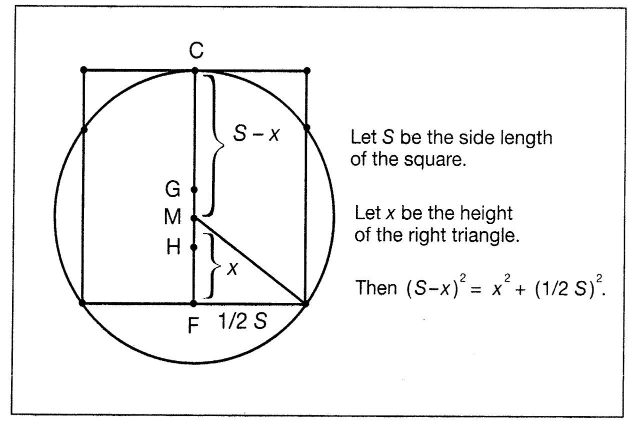


Fig. 19.10. One method by students, using the Pythagorean theorem

After they presented their idea to the whole class, we asked the class to discover why it worked. Before long, several students saw how to use the Pythagorean theorem to prove that point *M* did have the desired property (i.e., that it was equidistant from the intersection points), but they still did not have a good sense of why it worked.

We were intrigued to observe that these students started with a "black box" method that just worked for no known reason; then moved to an analytic method that, in their opinion, failed to provide a clear explanation; and finally produced an intuitive combination of both.

Finally, one student who had been using the "show grid" feature of The Geometer's Sketchpad reported with excitement that she had found a "3-4-5" right triangle, as shown in figure 19.11. On seeing this, the students realized why this method works. Assuming a unit square, because the lower side is split in half, each half is four-eighths of a unit long, whereas segment *CF* is divided such that point *M* is three-eighths of a unit from the bottom. This construction creates a right triangle with legs of three- and four-eighths, so the hypotenuse is five-eighths, matching the length of segment *CM*.

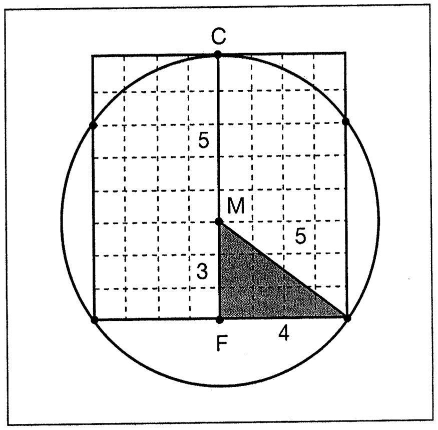


Fig. 19.11. Another method by students, using the Pythagorean theorem and showing an underlying 3-4-5 right triangle

Concluding Remarks

The investigation of intersections of a circle and a square served as a rich context for our students to engage in the process of doing geometry, and we encourage you to try it with your students. More important than using this particular task, however, is the opportunity it presents to help us create other rich tasks. We did not design the task to present or apply a particular concept or theorem, but rather as a means to guide students though the processes of representing, defining, classifying, constructing, and proving. The general structure of the investigation followed a *launch, explore,* and *extend* format, which transfers to other strands of mathematics education besides geometry.

We *launched* the investigation by beginning with a simple question and imprecise representational tools. From the beginning, students were able to take ownership of the investigation because the problem was posed in an open-ended

way. They were able to generate a wealth of configurations and were encouraged to decide for themselves whether any two configurations were the same. The initial question was deliberately imprecise in that we did not tell them what we meant by "a different way." Consequently, our students had to engage in the processes of defining and classifying as they explored the situation.

Secondly, as our students *explored* the situation more deeply, we expanded the initial investigation by encouraging them to attend to relationships and make conjectures concerning the situation. Exploring these conjectures led some students to see shortcomings in their initial definitions, which resulted in their decision to refine their classification schemes. They also began to use more-precise representational tools, such as The Geometer's Sketchpad. Doing so helped them connect the relationships they found with other concepts, such as the inscribed angle theorem. Thus, the problem, although initially presented in an intuitive manner, was open to more-precise analysis and connected with meaningful mathematics.

Furthermore, creating their own classification system for the circle-and-square intersection situation also helped our students see the need for more-formal justifications of their results. Our students were willing to *extend* the investigation by examining how the configurations they discovered could be constructed, and they were able to do so in multiple ways. The students were able to prove their constructions through other relationships, such as the Pythagorean theorem. Their proofs not only verified their results but also established connections with different concepts, an important role for proof in geometry (de Villiers 2003). When we designed the circle-and-square investigation, we intended it to begin informally and lead to formal constructions and proofs in a natural way.

Variations on this problem can lead to similarly rich geometric investigations. One might explore how other planar shapes, such as a square and an ellipse, intersect. Another would be to explore how a cube and a plane intersect in space. The amount of interesting mathematics that can be generated from such seemingly simple questions is surprising. Indeed, the theory of conics is historically related to a similar question concerning the intersection of a plane and a double cone. In sum, if instructors are willing to embark on a mission of exploration with their students with rich tasks such as these, then they are inviting students to "do geometry" with them.

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Geometer's Sketchpad files that support this article are found on the CD-ROM disk accompanying this Yearbook.