

Starry Night



Photograph 1
James Yamada's
Our Starry Night

PHOTOGRAPH BY BRIGITTE BENTELE; ALL RIGHTS RESERVED

"Mathematical Lens" uses photographs as a springboard for mathematical inquiry. The goal of this department is to encourage readers to see patterns and relationships that they can think about and extend in a mathematically playful way.

Edited by **Ron Lancaster**
ron2718@nas.net
University of Toronto
Toronto, Ontario M5S 1A1
Canada

Brigitte Bentele
brigitte.bentele@trinityschoolnyc.org
Trinity School
New York, NY 10024

At first glance, James Yamada's *Our Starry Night*, on view at the Doris C. Freedman Plaza in New York City from April to October 2008, is a black, twelve-foot-tall sculpture with 1900 unlit colored LED lights on its two flat surfaces (see **photograph 1**). When one passes through the sculpture's rectangular passageway, however, patterns of light appear. The explanation, on a nearby plaque and the Public Art Fund Project Web site, states: "When visitors walk through the portal in the piece, they trigger a metal detector hidden inside the structure's casing. This activates the LED lights that perforate

the exterior of the sculpture. Common everyday metal objects such as cell phones, keys, belts, jewelry, cameras, computers, and the like will trigger the lights; the luminosity and the light patterns seen in the piece will correspond to the quantity of metal detected. *Our Starry Night* is literally activated by the public, reinforcing the notion that art—and particularly public art—is dependent on the people around it." Watching the colored patterns appear led Brigitte Bentele to consider some mathematical questions. She asked a friend to help by walking through the passageway numerous times.



Photograph 2

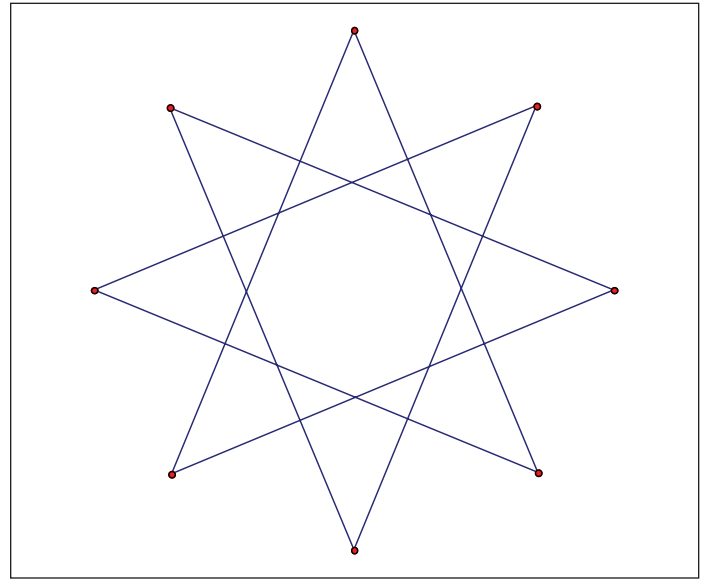
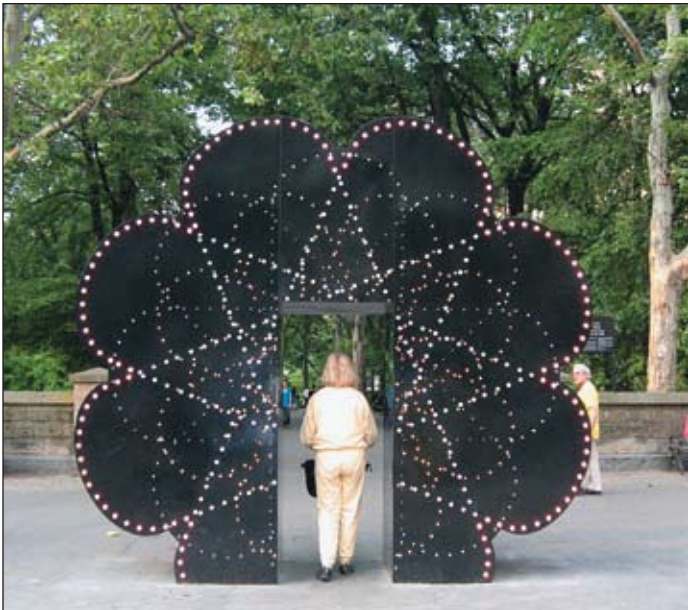


Fig. 1 Schematic diagram of the 8-pointed star



Photograph 3

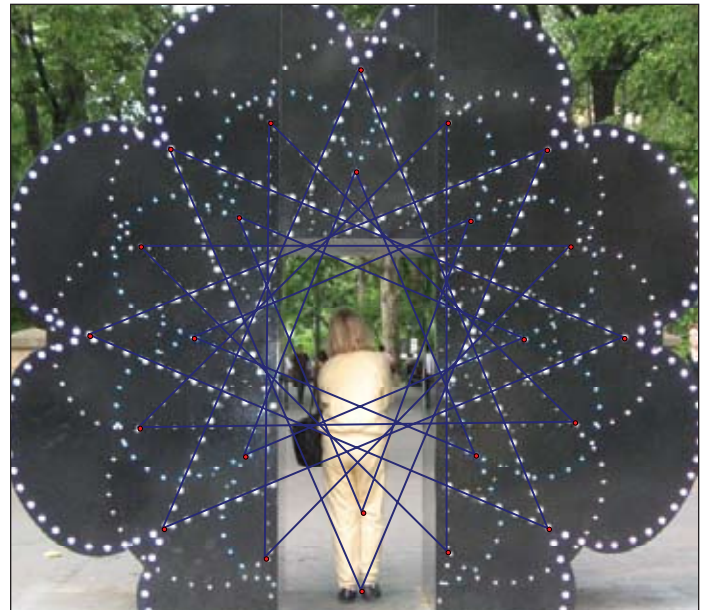


Fig. 2 All three 8-pointed stars superimposed on the photograph

PHOTOGRAPHS BY BRIGITTE BENTELE; ALL RIGHTS RESERVED

1. (a) One of the patterns revealed an 8-pointed star (see **photograph 2**), of which only 7 points are visible (the 8th point would be located where the passageway cuts through the sculpture). For the moment, assume that the 8th point is visible (see **fig. 1**).
- (b) Suppose that the artist decided to place 28 lights along each of the longest segments while placing only 1 light at each intersection of two segments. How many lights would he need?
2. Consecutive vertices of the star are connected by arcs that appear to be elliptical. The two segments at each vertex are extended before the ellipse is formed. There are 17 lights along each of the elliptical paths that surround the sculpture and 2 additional lights along the extended segments. Each vertex is shared by two extended arcs. How many additional lights are needed for the exterior arcs that light up when the star lights up?
3. Depending on the amount of metal on the person walking through the passageway, two more stars with elliptical extensions in different colors might appear. Both are smaller than the original star. The larger one has its vertices rotated by a 45° angle from the original, while the smaller one does not (see **photograph 3** and **fig. 2**).
- (a) If the ratio of the dimensions of the three stars is 8:7:6, how many lights would each of the two smaller stars need?
- (b) How many additional lights are needed for the other arcs?

- One vertex of the largest star is not visible in the sculpture. Suppose that the two lowest vertices are 20 in. above the ground and that the distance between them is 78 in. Would the missing vertex be above or below the ground if the sculpture were solid?
- Equations for ellipses can be given using x - and y -coordinates in standard form or through trigonometric functions in parametric form. The sculpture's six half ellipses that are visible rotate around the center of the star. In standard form, rotated ellipses have an x - y term. Parametric form can be obtained for a rotated ellipse by using the following matrix:

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

The lengths of the major and minor axes of the ellipses are 44.5 in. and 33 in., respectively. Further, if one looks closely, one will see a small distance between the vertices of the major axes; this distance measures 4.75 in. With the origin at the center of the rotation, write parametric equations for the half ellipses that create the sculpture's border.

- A Geometer's Sketchpad representation of the sculpture (see **fig. 3**) ignores the doorway and the ground, which cuts off the sculpture. Using the measurements given earlier, find the sculpture's surface area.

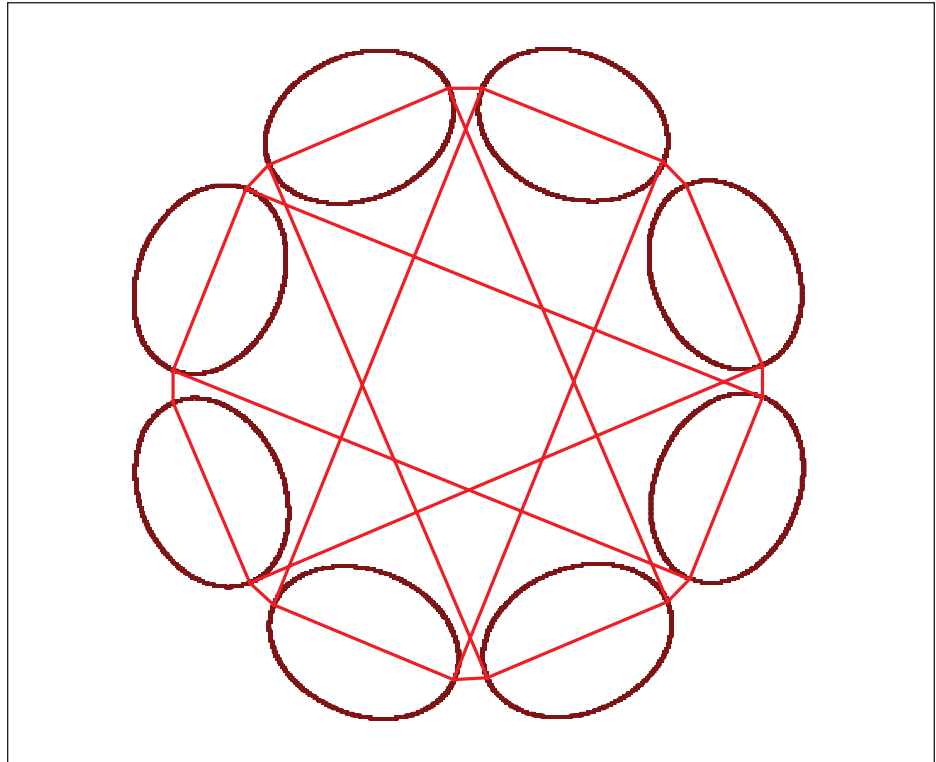


Fig. 3 GSP schematic representation of the sculpture

- Ignoring the doorway and the ground, both of which eliminate part of the design, makes the mathematics simpler. This question takes into account the doorway and the ground but asks for an estimate rather than an exact answer. Inside the sculpture are the mechanisms that set off the lights as one walks through the doorway. The doorway measures 32 in. wide and 78 in. high, and the sculpture is 40.5 in. deep. Using these dimensions with

those already provided, estimate the sculpture's volume.

The editors returned to photograph the sculpture at night (see **photographs 4 and 5**). We noticed that even when no one was in the doorway, the lights would sometimes change, perhaps responding to the wind. The many visitors to the sculpture delighted in having their pictures taken while the sculpture entertained them with its light show.



PHOTOGRAPHS BY BRIGITTE BENTELE; ALL RIGHTS RESERVED

MATHEMATICAL LENS solutions

1. The artist requires $28 \times 8 - 24 = 200$ lights. One way to arrive at this result is to count the number of intersections and subtract this from the number of lights on each of the 8 longest segments of the star (do not forget to count the vertices themselves). The number of intersections can be found without counting each one:

$$\frac{8 \times 6}{2} = 24$$

Each long segment intersects with 6 of the other segments (each long segment is parallel to one other segment; we divide by 2 to avoid double-counting points).

2. The 2 lights at the extensions of the sides are the vertices of the star and have already been counted, so each arc has only 2 additional lights. Thus, the total of additional lights is $19 \times 8 = 152$.
3. (a) Using the ratio provided, we obtain $8:7:6 = 200:175:150$.
- (b) The same ratio yields the equation $8:7:6 = 152:133:114$.
4. The missing vertex would be above the ground. Connecting the eight consecutive vertices of the star forms an octagon. Angle AOB is an interior angle of the octagon (see **fig. 4**), so its

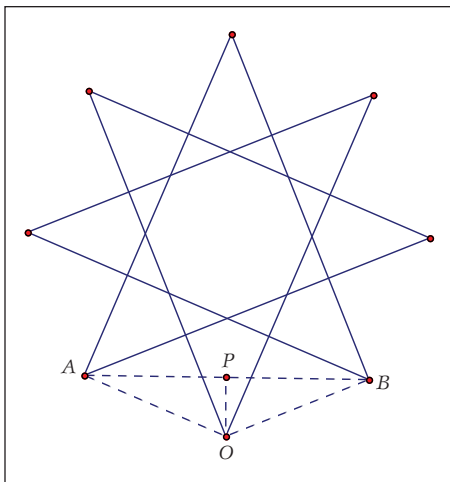


Fig. 4 Determining the location of the "missing" vertex

measure is 135° . Connecting vertices A and B with the missing vertex O would form isosceles triangle AOB . The horizontal distance AB is 78 in. Therefore, the vertical distance OP can be found by evaluating $39 \tan(22.5^\circ) \approx 16.154$ in. Since the two vertices are 20 in. above the ground, this missing vertex is about 3.846 in. above the ground.

5. First, we are given a set of parametric equations for an ellipse that has not rotated from the horizontal. The x -coordinate of the parametric equations is given by $22.25 \cos(t)$, where 22.25 is half the length of the major axis. The y -coordinate needs to be shifted by a vertical distance of length OC (see **fig. 5**). To find the distance OC , we first find the length of OD . Since triangle BOF is isosceles with central angle BOF of 45° , $OD = 22.25 \tan(67.5^\circ)$. CD can then be found by determining AB . AG is half the distance between the vertices of consecutive ellipses, so $AG = 2.375$ and

$$AB = \frac{2.375}{\cos(67.5^\circ)}$$

Therefore, the y -coordinate of the nonrotated ellipse is

$$16.5 \sin(t) + 22.25 \tan(67.5^\circ) + \frac{2.375}{\cos(67.5^\circ)}$$

The other four sets of equations graph the right portion of the elliptical halves if a window where t varies from 0° to 180° is set.

$$\begin{aligned} x_{1T} &= 22.25 \cos(t) \\ y_{1T} &= 16.5 \sin(t) + 22.25 \tan(67.5^\circ) + 2.375 / \cos(67.5^\circ) \\ x_{2T} &= x_{1T} \cos(22.5^\circ) + y_{1T} \sin(22.5^\circ) \\ y_{2T} &= -x_{1T} \sin(22.5^\circ) + y_{1T} \cos(22.5^\circ) \\ x_{3T} &= x_{2T} \cos(45^\circ) + y_{2T} \sin(45^\circ) \\ y_{3T} &= -x_{2T} \sin(45^\circ) + y_{2T} \cos(45^\circ) \\ x_{4T} &= x_{3T} \cos(90^\circ) + y_{3T} \sin(90^\circ) \\ y_{4T} &= -x_{3T} \sin(90^\circ) + y_{3T} \cos(90^\circ) \\ x_{5T} &= x_{4T} \cos(135^\circ) + y_{4T} \sin(135^\circ) \\ y_{5T} &= -x_{4T} \sin(135^\circ) + y_{4T} \cos(135^\circ) \end{aligned}$$

6. The surface area can be found by finding the area of the 16-gon and adding the area of the 8 half ellipses. (The polygon has 16 sides since the segments of the 8-pointed star are extended before the vertices are connected.) To find the area of the polygon, consider **figure 5**. We find the area of region $ACOG$ and multiply by 16. We divide region $ACOG$ into triangles AGB and BOD and rectangle $ABDC$. We know that $BD = 22.25$ and, from the answer to question 5, that $OD = 22.25 \tan(67.5^\circ)$.

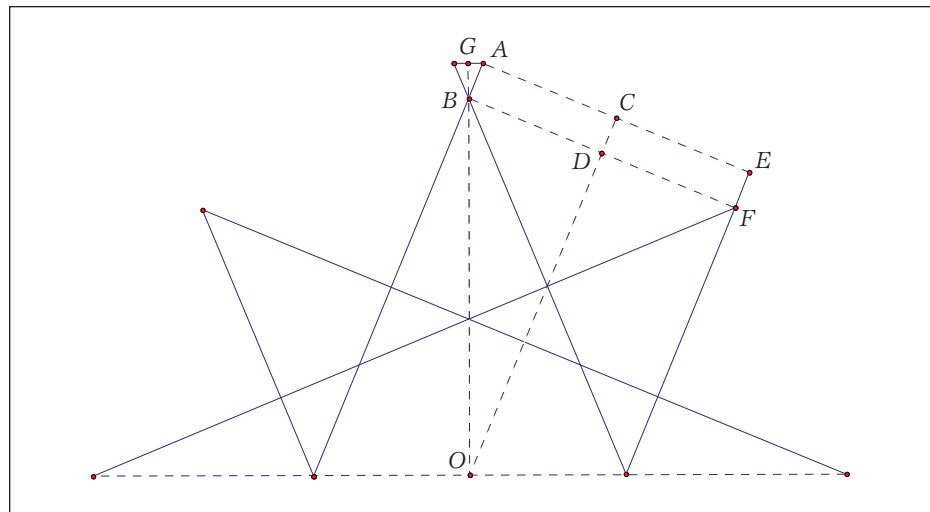


Fig. 5 Determining the surface area of the sculpture

Therefore, the area of $\triangle BOD = (1/2)(22.25)^2 \tan(67.5^\circ) \approx 597.593 \text{ in}^2$. We know that

$$AB = \frac{2.375}{\cos(67.5^\circ)}$$

Therefore,

$$\begin{aligned} \text{the area of rectangle } ABDC &= 22.25 \times \frac{2.375}{\cos(67.5^\circ)} \approx 138.09 \text{ in}^2. \end{aligned}$$

Finally, using the formula for the area of a triangle in terms of two sides and the sine of the included angle, we see that

$$\begin{aligned} \text{the area of } \triangle AGB &= \frac{1}{2}(2.375) \left(\frac{2.375}{\cos(67.5^\circ)} \right) \sin(67.5^\circ) \\ &= \frac{1}{2}(2.375)^2 \tan(67.5^\circ) \\ &\approx 6.81 \text{ in}^2. \end{aligned}$$

The total area of the polygon is $16 \times 742.49 = 11,879.84 \text{ in}^2$. The formula for the area of an ellipse is $\text{area} = \pi ab$, where a and b are half the length of the axes of the ellipse. Therefore, the area of a half ellipse is $(1/2)\pi(22.25) \cdot (16.5) \approx 576.68 \text{ in}^2$. Multiplying this by 8, we find that the total area of the elliptical regions is $4,613.43 \text{ in}^2$. The total surface area of the face of the sculpture as described above is about $16,493.27 \text{ in}^2$, or about 114.54 ft^2 .

7. From the surface area found in the answer to question 6, we first subtract the area of the door, which is $32 \text{ in.} \times 78 \text{ in.} = 2496 \text{ in}^2$. **Figure 6** shows the **figure 3** sketch superimposed on the photograph of the sculpture. It appears that approximately the equivalent of one ellipse is lost below the ground, or approximately 1153 in^2 . Therefore, the total volume can be computed as follows: Volume = (surface area found in question 6 – area of passageway – area lost under the ground) \times (depth of the sculpture) = $(16,493.27 - 2496 - 1153) \times (40.5) \approx (12,844) \times (40.5) \approx 520,182 \text{ in}^3$, or about 301 ft^3 . ∞

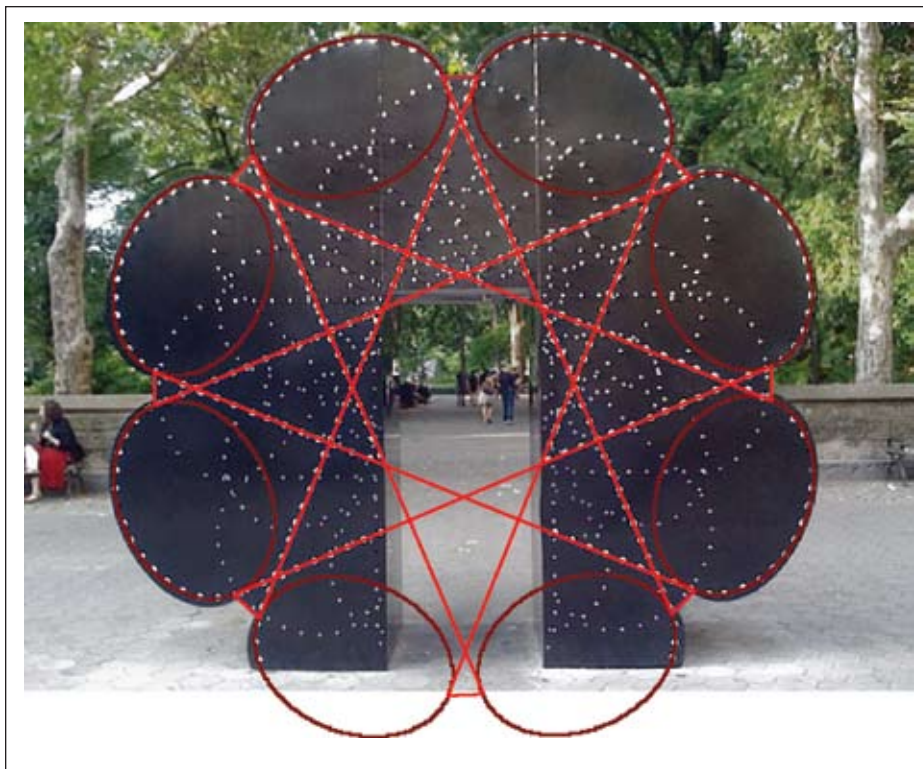


Fig. 6 The schematic representation superimposed on the photograph of the sculpture

PHOTOGRAPH BY BRIGITTE BENTELE; ALL RIGHTS RESERVED

Members who wish to use this month's photographs in a classroom setting can download the images from NCTM's Web site, www.nctm.org. Follow links to *Mathematics Teacher*, and choose Current Issue. Then select Mathematical Lens from the Departments, and look for the link to the image.

Send Us Your Mathematical Photographs!

Have you ever seen a building, a bridge, a sign, or a natural phenomenon that stimulated mathematical thoughts? Why not take a photograph and send it to NCTM, along with the mathematical questions that the photograph inspires? The questions can be playful, imaginative, curious, and inventive; they can also be mathematical extensions sparked by the photograph.

If the photograph includes identifiable people, the photographer must obtain signed release forms. Photographers must also obtain release forms if trademarked items are shown. Original photographs must be supplied either as hard copy or digitally as 300 dpi images in .jpg format. For details on releases and digital standards, please see the NCTM Web site. Photographs will not be returned.

Send the photographs, diagrams, list of questions, solutions, and completed release forms to the "Mathematical Lens" editors.

