TAKE TIME FOR ACTION

An Assessment of Students'

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HOW WELL DO YOU THINK YOUR STUDENTS understand the underlying concepts related to fractions, angles, probability, or any other complex ideas that we attempt to teach? Sometimes students' answers on tests fail to communicate what they really understand about a concept. If you wanted to assess your students' understandings of the *angle concept*, what kinds of questions could you pose to reveal their true thinking?

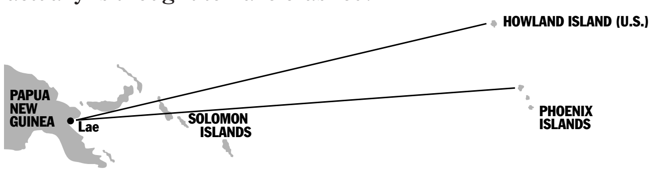
Our students had just completed the "Shapes and Designs" unit from the *Connected Mathematics Project (CMP)* (Lappan et al. 1996) that had provided a wide variety of explorations of two-dimensional geometry, including the angle concept. To assess this unit, we asked students to participate in two different activities. First, we asked them to write a definition of *angle,* a more difficult task than it would seem. A concept as abstract as *angle* is difficult to capture with just a few phrases. Second, we decided that individual students' struggles are sometimes the best way to identify misunderstandings of a group. We presented our students with comments made by other students while studying the same *CMP* unit (Keiser 2000). They considered the comments, decided if they agreed or disagreed, and explained why.

Before you read about our research, ask your students to respond to two of the situations in **figure 1.** Did misconceptions arise about the ideas of angle? Did you find their responses surprising? In what ways?

Directions: Write a definition af ANGLE in your own words. Make it as complete as possible. You may use more than one or two sentences.

In Situations 1–3, other sixth-grade students discussed their ideas about angles. Read the situations carefully, and explain whether you "agree" or "disagree" and why. If you are unsure, answer with the response that *best* describes which way you are leaning.

Situation 1: Amelia Earhart, a pioneer in aviation, was lost in a plane crash because of a navigation error when she was flying around the world. Her navigator made a 7.5 degree error. The illustration below shows both her real destination and where she actually is thought to have crashed.



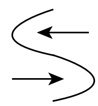
Adapted from Small Blue Planet. © Copyright 1993–1995 by Now What? Software, San Francisco. The original map was created from maps made by the Central Intelligence Agency and the Defense Mapping Agency.

The teacher asked, "Was she off very much as far as the angle is concerned?" and most of the class said, "No." But Claire argued, "But that's like on a small scale though. In real life it would be really big." John agreed and added, "It would be like 200 degrees." The teacher asked, "What do you mean by that?" and Claire answered, "That's a small map. If you made the map bigger, if you blew it up bigger, it would be a different distance, because [the islands] wouldn't be so scrunched together. It would be farther apart. And if it was real life, it would be a really big angle."

Do you agree or disagree with Claire?

Situation 2: Students were looking around their classroom to find examples of angles, and Cindi said, "Can you find angles in any letter of the alphabet? Except for 'S' because it's kind of, well . . . maybe it does have angles! Because you know how S's sort of have corners like here and here?" She draws an S on the board (see below) and points to two places. "So, maybe it works for all letters in the alphabet?" Dan said, "Except O." Cindi agreed, "Yeah, except O."

Do you agree or disagree with Cindi (that S's have angles)?



Situation 3: One student in the sixth-grade class had a lot of questions about 0 degree, 180 degree, and 360 degree "angles." A few of her comments are below.

1. "I have a question about a 360 degree angle. Isn't an angle like a point on something? I'm kind of curious about 180 degrees, too. But 360 degrees, all it is, is a circle. So I don't really think . . . so I am kind of wondering why 360 degrees and 180 degrees are considered angles?"
2. "Because I saw an angle as having a vertex point and then two lines going different ways. And I know [the rays] are going different ways [in a 180 degree angle], but you can't tell where the vertex is and how they are going different ways or if they're sort of doing the same thing."
3. "And all I think what the angle really is, is two lines put together on their corner, propped up on their corner, but this has no corner to be on. I don't understand how an angle can just be a turning motion and not two lines."

Do you agree or disagree?

Fig. 1 Explaining the meaning of "angle"

Understanding of Angle

Discussion of the Tasks

WE USED BOTH TYPES OF ASSESSMENTS, WRITING A definition and the situations in **figure 1,** as a way to explore our students' *concept images* of angle. Tall and Vinner (1981) first defined *concept image* to "describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Primarily, we were hoping to challenge the students to think critically about the concept so that we could identify interesting questions and possible misconceptions that our students still had about *angle.* Then, we could follow up our action research by having a class discussion to address these questions.

Our 77 sixth-grade students came from three different classes and had just completed the "Shapes and Designs" unit the day before. During one class period, we first asked them to write a definition for angle, which we collected before we presented them with two of the three situations given in **figure 1.** (We only asked them to complete two of the three because of time constraints, although asking students to react to all three would have been interesting, as well!) The three situations reveal either misunderstandings of certain features of an angle or contain examples of other students' struggles to make sense of a certain idea. In their reactions, we encouraged students to be complete in describing *why* they agreed or disagreed with the comments.

Classification of Students' Definitions

IF YOU ASKED YOUR STUDENTS TO WRITE A DEFINITION for *angle,* what definitions do you think they would give? Would their definitions include all the items that you consider to be an angle? Or would certain features be emphasized, thereby excluding certain kinds of angles, such as reflex angles or angles with measures of 0, 180, or 360 degrees? Would they focus more on the inclination of the rays, on the interior, or on the turning motion from one ray to another?

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The truth is that all definitions, by their very nature, tend to limit concepts because they establish boundaries. Consider, for example, this definition of angle: "An 'angle with vertex A' is a point A together with two distinct nonopposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the sides of the angle) emanating from A" (Greenberg 1993). This definition from a well-known college geometry textbook excludes 0, 180, and 360 degree angles, and does not distinguish between an x degree angle and a $(360 - x)$ degree angle. Instead, it focuses

TABLE 1
Students' Definitions

Emphasis of Definition	Number of Students with This Emphasis (N = 78)	Percentage of Students
The degrees or the measure itself "An angle is the number of degrees in a corner."	23	29.5%
The line segments that meet "An angle is where two line segments intersect to form an angle."	20	25.6%
The opening "How big apart the two lines are apart where the vertex is."	6	7.7%
The point "An angle is where two vertices meet and make a point."	6	7.7%
Measure of the edge "An angle is the measure of one side of a shape."	4	5.1%
Combination of two of these "I think an angle is the measure in degrees between two line segments that are touching."	8	10.3%
Vague or wrong statements "An angle is a shape that has straight lines and at least 3 angles."	11	14.1%

TABLE 2
Analyzing Situations 1, 2, and 3

	Number of Students Who Agreed	Number of Students Who Disagreed	Percentage of Students Who Agreed	Percentage of Students Who Disagreed
Situation 1	42	13	76%	24%
Situation 2	12	24	33 1/3%	66 2/3%
Situation 3	3	17	15%	85%

primarily on the two rays and gives no indication of rotation from one ray to the other, the angle's interior, or of how the angle is measured. There are actually very good reasons to constrain the angle definition in this way, one of which is that many of the angle theorems proven in the textbook do not pertain to angles measuring 180 degrees and larger; therefore, it is easier to exclude them rather than treat them as exceptions over and over again. However, when students make exclusions or focus too heavily on a certain aspect, they may be communicating that they do not have a well-developed concept image, one that is flexible enough to be applied to many different situations.

Two of us read through our students' definitions and classified them by the emphasis in the definition. For example, if the students wrote, "The angle is the vertex of a polygon," we classified that definition as one that emphasizes *point,* since the vertex is the primary emphasis (see **table 1**). We noticed right away that two emphases were most likely to be made—an emphasis on the measure of the angle itself and an emphasis on the intersection of line segments. Some less common definitions emphasized the opening between the rays or the vertex itself. Eight students wrote what we classified as "combination" definitions in that these combined two or more of the features necessary for fully understanding the angle concept. However, eleven students wrote definitions that were either too vague to understand or were simply incorrect. Four other students wrote such definitions as "An angle is the measure of one side of a shape," and we are investigating further to determine whether these students simply were confused in their use of vocabulary or if they really believed an angle contains only one side.

We were surprised to see that so many of our students defined the angle *by* its measure. Students do not often make this kind of mistake with other attributes. For example, a line segment is not often confused with measures of its length, such as inches, feet, and so on. Maybe this situation occurs because length is measured in so many different units, whereas angles are only measured in degrees and in radians, and our students have not yet been exposed to radian measure of angles. Thus, perhaps it is difficult for students to recognize that an angle is not the same as the measure of that angle.

At this stage in their development, the activity of formulating usable definitions is probably too formal for our students, but the discussion that results can challenge students to think critically about the words chosen as well as consequences that can occur when one aspect is emphasized too heavily. If you asked your students to write definitions for *angle,* do you think you would get similar results? Would the discussion that followed be educational for you and your students?

A Look at the Problem Situations

SINCE THE PROBLEMS PRESENTED IN **FIGURE 1** WERE situations that posed great struggles for students who were in the midst of learning from the "Shapes and Designs" unit, we were curious if some of the problems might be less troublesome for our students who had completed the unit. As **table 2** indicates, the majority of the students did not struggle with situations 2 and 3. The experiences from "Shapes and Designs," along with their past experiences, may have helped many of our students to eliminate the curved lines from their concept image of angle. Our students also did not seem to have trouble visualizing 0 degree, 180 degree, or 360 degree angles even though these special angles do not look like "corners," and they are best visualized by thinking of the turning motion of the rays instead of focusing on the rays themselves. In fact, one student wrote, "I disagree with [the] person because a 360 degree turn isn't a circle. It just is the measure from a point all the way around to meet itself at the other side." (Note that this student also is talking about a point, rather than a ray, that is in motion. Even though this student seems to be able to visualize a 360 degree rotation, he lacks the vocabulary to properly express his thinking.)

With situation 1, however, our students struggled to see that if an angle is "scaled up," or enlarged, so as to maintain similarity among figures, the measure of the angle remains constant. Of the students who read situation 1, 76 percent agreed with Claire that the angle drawn in their books, from which the diagrammatic map in situation 1 was adapted, would be smaller than the "scaled up" version from real life. One student wrote, "Yes [I agree], the map is a small version even though it may look like [an] actual picture. If you blew it up, it would be a bigger angle." Another wrote, "I agree because a bigger picture is a bigger degree."

These misconceptions can be explained in a number of ways. If students see an angle as the linear distance between two rays or as the area between two rays, it is logical to conclude that the further one moves away from the vertex of the angle, while staying in the interior of the angle, the larger the angle seems to be. Students who believe an angle has to do with the length of the rays also would conclude that a scaled-up angle would have a larger measure. These misconceptions might not be adequately addressed until students have a better understanding of similarity. When they explore similar figures, they will learn that although all the side lengths increase, the angle measures stay constant. The misconception also points to a larger problem for many students—that of proportional reasoning. When some students learn to use scale factors for increasing or decreasing the size of figures, they often believe that all attributes increase or decrease in the same way. For example, if a triangle is "doubled" in size, that terminology can be confusing: in doubling the side lengths, the perimeter is doubled, the area is quadrupled, but the angles remain constant!

Implications for Teaching and Learning

THIS SMALL COLLECTION OF DATA FROM OUR STUDENTS gave us great insight into their thinking. We learned that we need to think of a way to emphasize the difference between an attribute and its measure. We learned that many of our students still lack geometric vocabulary to express what they mean. We also learned that the constancy of the angle measure as an angle is enlarged or reduced is an idea that few of our students really understand. We also discovered that having students react to other students' comments is a very nonthreatening way to assess their thinking. Our students liked hearing that these comments came from students in another state almost eight years ago, and yet the ideas expressed by both groups of students were very similar. Using anonymous problem situations seems to be a good strategy for generating discussions. Most important, we learned that asking our students questions in a nonthreatening, nongraded manner is fruitful not only for the students but also for our own growth as reflective practitioners.

References

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